1	2	3	4	5	6	7	8	9	10
b	c	b	a	d	d	d	d	a	a
11	12	13	14	15	16	17	18	19	20
b	c	a	c	b	c	d	a	b	d
21	22	23	24	25					
c	a	b	c	c					

1. 在數 $2012! = 1 \cdot 2 \cdot \cdots \cdot 2011 \cdot 2012$ 的後頭有多少個連續的零?
   How many zeros are there at the end of the number $2012! = 1 \cdot 2 \cdot \cdots \cdot 2011 \cdot 2012$?

   Solution: Each occurrence of $5^1 = 5$ contributes one zero; each occurrence of $5^2 = 25$ contributes two zeros; each occurrence of $5^3 = 125$ contributes three zeros; and each occurrence of $5^4 = 625$ contributes four zeros. There are $\lfloor \frac{2012}{5} \rfloor + \lfloor \frac{2012}{25} \rfloor + \lfloor \frac{2012}{125} \rfloor + \lfloor \frac{2012}{625} \rfloor = 402 + 80 + 16 + 3 = 501$ zeros at the end of 2012!.

   (a) 622
   (b) 501
   (c) 335
   (d) 278
   (e) 以上皆非 None of the above

2. 小於 100 的質數裡, 其位數和為 10 的質數有幾個?
   The number of prime numbers less than 100 which have digits that sum to 10 is?

   Solution: Of the two digits integers whose digits sum to 10 and not even are: 19, 91, 37, 73, 55. Among those, $91 = 7 \times 13$ and $55 = 5 \times 11$ are not a prime number.

   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) 以上皆非 None of the above

3. 將數列 $1, 2, 3, \ldots, 2010, 2011, 2012$ 裡的所有位數的加起來, 亦即 $1 + 2 + 3 + \cdots + (2 + 0 + 1 + 0) + (2 + 0 + 1 + 1) + (2 + 0 + 1 + 2)$, 答案為何?
Consider the sequence $1, 2, 3, \ldots, 2010, 2011, 2012$. What is the sum of all the digits of the numbers in it? That is, what is $1 + 2 + 3 + \cdots + (2 + 0 + 1 + 0) + (2 + 0 + 1 + 1) + (2 + 0 + 1 + 2)$?

Solution: Note that $1+9+9+9 = 1+(1+9+9+8) = 2+(1+9+9+7) = \cdots = (1+0)+(1+9+8+9) = \cdots = (9+9+9)+(9+9+9)$. Thus, the sum of the digits in the numbers $1, \ldots, 1999$ is $(\frac{1998}{2} + 1) \cdot (1 + 9 + 9 + 9) = 1000 \cdot 28 = 28000$. Now, the sum of the digits of $2000, \ldots, 2012$ is $(2+0+0+0)+(2+0+0+1)+\cdots+(2+0+0+9)+(2+0+1+0)+(2+0+1+1)+(2+0+1+2) = (2 + \cdots + 11) + 3 + 4 + 5 = 77$. Therefore, the answer is 28077.

(a) 10021
(b) 28077
(c) 34265
(d) 39941
(e) 以上皆非 None of the above

4. 在一個六邊形 $ABCDEF$ 的六個邊分別塗上紅、黃、藍顏色。讓兩個相鄰邊是不同色的塗法有多少種?
The edges of the hexagon $ABCDEF$ are each to be colored either red, blue, or green. How many distinct colorings are possible if no two adjacent edges may have the same color?

Solution: This is equivalent to ask how many ways there are to make six-letter words from the letters $\{a, b, c\}$ so that no two consecutive letters are the same while the first and the last letters are different. There are $3 \cdot 2^5 = 96$ words with no two consecutive letters are the same. Among them are those which has the same the first and last letters. Consider first words of the form aXa, where X is one of $babc$, $bacb$, $bcab$, $bcac$, $bcbc$, $cbab$, $cabc$, $cbac$, $cbab$, $cbcb$. There are also 10 words of the form bXb, and 10 of the form cXc. Take away these from the 96 words to get the answer: 66.

(a) 66
(b) 55
(c) 42
(d) 30
(e) 以上皆非 None of the above

5. 對每一個正整數 $k=1,2,\ldots,100$, 考慮多項式 $(1+x)^k(1-x)^{100-k}=1+a_kx+b_kx^2+c_kx^3+\ldots$。係數 $b_1,b_2,\ldots,b_{100}$ 中有幾個是正數?
For positive integer k, $1\le k\le 100$, consider the polynomial $(1+x)^k(1-x)^{100-k}=1+a_kx+b_kx^2+c_kx^3+\ldots$. How many of the numbers $b_1,b_2,\ldots,b_{100}$ are positive?

Solution: From
$$(1+kx+\frac{1}{2}k(k-1)x^2+\ldots)(1-(100-k)x+\frac{1}{2}(100-k)(100-k-1)-\ldots)$$
we see that the coefficient b_k of the term x^2 is $-k(100-k)+\frac{1}{2}k(k-1)+\frac{1}{2}(100-k)(99-k)=2(k-45)(k-55)$. Thus, b_k is positive when $k<45$ and $k>55$. That is, there are $44+45=89$ of them positive.

(a) 39
(b) 50
(c) 75
(d) 89
(e) 以上皆非 None of the above

6. 假設 a 和 b 是整數, 且多項式 $5x^4-4x^3+3x^2-ax+b$ 可以被 x^2+1 整除。請問 $a-b$ 可能是下列何者?
Let a and b be integers such that the polynomial $5x^4-4x^3+3x^2-ax+b$ is divisible by x^2+1. What could be the value of $a-b$?

Solution: Divide x^2+1 into $5x^4-4x^3+3x^2-ax+b$ to get reminder $(4-a)x+(2+b)$. This has to be zero. Hence $a=4$ and $b=-2$.

(a) -6
(b) -2
(c) 2
(d) 6
(e) 以上皆非 None of the above

7. 假設 $\frac{87}{17}=w+\frac{1}{y+\frac{1}{x}}$, 其中 x、y、w 為正整數,$x+y+w$ 之值為何?
If $\frac{87}{17}=w+\frac{1}{y+\frac{1}{x}}$, where x, y, and w are all positive integers, then what is the value of $x+y+w$?

Solution: $87 = 5 \cdot 17 + 2$, and so $w = 5$; $17 = 2 \cdot 8 + 1$, and so $y = 8$ and $x = 2$.

(a) 21
(b) 19
(c) 17
(d) 15
(e) 以上皆非 None of the above

8. 假設函數 $f(x)$ 滿足對任意實數 m、n, $f(mn) = f(m-n)$。如果 $f(4) = 3$, 則 $f(-2) + f(6)$ 為何?
Consider the function $f(x)$ such that $f(mn) = f(m-n)$ for all real numbers m and n. If $f(4) = 3$, find $f(-2) + f(6)$.

Solution: $f(0) = f(2-2) = f(2 \cdot 2) = f(4) = 3$. Hence $f(r) = f(r-0) = f(r \cdot 0) = f(0) = 3$ for all real number r.

(a) 0
(b) 2
(c) 3
(d) 6
(e) 以上皆非 None of the above

9. 我們定義一個運算 $a \star b$ 的結果為 a 和 b 兩個數裡大的那一個, 並且定義 $a \star a = a$。再定義一個運算 $a \circ b$ 的結果為 a 和 b 兩個數裡小的那一個, 並且定義 $a \circ a = a$。如果 a、b、c 是相異三個數, 且 $a \circ (b \circ c) = (a \circ b) \star (a \circ c)$, 則:
Let $a \star b$ represent the operation on two numbers a and b, which selects the larger of the two numbers, with $a \star a = a$. Let $a \circ b$ represent the operation which selects the smaller of the two numbers with $a \circ a = a$. If a, b, and c are distinct numbers, and $a \circ (b \circ c) = (a \circ b) \star (a \circ c)$, then we have:

Solution: If b is the smallest, then $a \circ (b \circ c) = b$ while $(a \circ b) \star (a \circ c) = a \circ c$ which is not b. Thus, b cannot be the smallest. By symmetry, c cannot be the smallest. Therefore, a has to be the smallest. This makes (b), (c) and (d) false. Also, both $b > c$ and $c > b$ do not affect the identity.

(a) $a < b$ and $a < c$
(b) $a > b$ and $a > c$
(c) $c < b < a$
(d) $c < a < b$
(e) 以上皆非 None of the above

10. 令 $f(x) = \frac{x-2}{x-1}$, $f_2(x) = f(f(x))$, $f_3(x) = f(f(f(x)))$, 依此類推. 則 $f_{101}(x)$ 為何?
Let $f(x) = \frac{x-2}{x-1}$, $f_2(x) = f(f(x))$, $f_3(x) = f(f(f(x)))$, etc. What is $f_{101}(x)$?

Solution: $f_2(x) = f(f(x)) = x$, $f_3(x) = f(x)$, etc.

(a) $\frac{x-2}{x-1}$
(b) $\frac{x^{101}-202}{x^{101}-101}$
(c) $\frac{101x-202}{101x-101}$
(d) x
(e) 以上皆非 None of the above

11. 如果 $\cos x + \sin x = \frac{1}{3}$, 則 $\cos^3 x + \sin^3 x$ 的值為何?
Suppose that $\cos x + \sin x = \frac{1}{3}$, what is the value of $\cos^3 x + \sin^3 x$?

Solution: $\cos^3 x + \sin^3 x = (\cos x + \sin x)(\cos^2 x - \cos x \sin x + \sin^2 x) = \frac{1}{3}(1 - \cos x \sin x)$. From $\frac{1}{9} = (\cos x + \sin x)^2 = 1 + 2\cos x \sin x$ we get $\cos x \sin x = -\frac{4}{9}$. Hence $\frac{1}{3} \cdot (1 + \frac{4}{9}) = \frac{13}{27}$ is the result.

(a) $\frac{12}{27}$
(b) $\frac{13}{27}$
(c) $\frac{14}{27}$
(d) $\frac{15}{27}$
(e) 以上皆非 None of the above

12. 下列哪個數與 $\sqrt{10+4\sqrt{6}} + \sqrt{10-4\sqrt{6}}$ 相同?
Which of the following is the value of $\sqrt{10+4\sqrt{6}} + \sqrt{10-4\sqrt{6}}$?

Solution: Note. $\left(\sqrt{10+4\sqrt{6}} + \sqrt{10-4\sqrt{6}}\right)^2 = 10 + 4\sqrt{6} + 2\sqrt{100 - 16 \cdot 6} + 10 - 4\sqrt{6} = 24 = \left(2\sqrt{6}\right)^2$.

(a) $2\sqrt{3}$
(b) $3\sqrt{2}$
(c) $2\sqrt{6}$
(d) 4
(e) 以上皆非 None of the above

13. 如果三個相異正實數 x、y、z 滿足下列等式, 則 $\frac{y}{x}$ 為何?
Suppose that x, y, and z are three distinct positive real numbers satisfying the following identities. What is $\frac{y}{x}$?

$$\frac{y}{x-z} = \frac{x+y}{z} = \frac{x}{y}$$

Solution: Assume that $\frac{x}{y} = k$, i.e. $x = ky$. Then $ky + y = kz$ and $y = (x-z)k = k^2y - kz = k^2y - ky - y$. Hence $k^2 - k - 2 = 0$. Solving it to get $k = -1$ or 2. As x and y are positive, $k = -1$ is not possible. Hence $\frac{y}{x} = 1/(\frac{x}{y}) = \frac{1}{k} = \frac{1}{2}$.

(a) $\frac{1}{2}$
(b) $\frac{2}{3}$
(c) 1
(d) $\frac{5}{3}$
(e) 以上皆非 None of the above

14. 定義數列 $a_1 = 1$, $a_2 = 2$, 及 $a_{n+2} = a_{n+1} + 2a_n$, $n \geq 3$。則 $a_{100} + a_{99}$ 為何?
Define a sequence by $a_1 = 1$, $a_2 = 2$, and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 3$. What is $a_{100} + a_{99}$?

Solution: $a_{n+2} + a_{n+1} = 2a_{n+1} + 2a_n = 2(a_{n+1} + a_n) = \cdots = 2^n(a_2 + a_1) = 3 \cdot 2^n$. Hence $a_{100} + a_{99} = 3 \cdot 2^{98}$.

(a) $\frac{99(99+1)}{2}$
(b) $\frac{100(100+1)}{2}$
(c) $3 \cdot 2^{98}$
(d) 100!
(e) 以上皆非 None of the above

15. 解聯立方程組:
Solve the system of equations:

$$\begin{cases} 2^x \cdot 3^x = 4, \\ x + y = 5. \end{cases}$$

Solution: $2\ln 2 = \ln 4 = \ln(2^x \cdot 3^y) = x \cdot \ln 2 + y \cdot \ln 3 = (5-y) \cdot \ln 2 + y \cdot \ln 3$, and so $-3\ln 2 = (\ln 3 - \ln 2)y$. This gives $y = \frac{3\ln 2}{\ln 2 - \ln 3}$. Similarly, $x = \frac{2\ln 2 - 5\ln 3}{\ln 2 - \ln 3}$.

(a) $x > 0$, $y < -5$

(b) $x = \frac{2\ln 2 - 5\ln 3}{\ln 2 - \ln 3}$, $y = \frac{3\ln 2}{\ln 2 - \ln 3}$

(c) $x = \ln 2$, $y = \ln 3$

(d) $x = \frac{\ln 2 - \ln 3}{\ln 5}$, $y = \frac{\ln 5}{\ln 2 + \ln 3}$

(e) 以上皆非 None of the above

16. 令 $S = \{5, 5^2, 5^3, \ldots, 5^{10}\}$. 假設 a 和 b 是由 S 中選取的相異數。有多少序對 (a, b) 讓 $\log_a^b$ 為整數?。
Let $S = \{5, 5^2, 5^3, \ldots, 5^{10}\}$. Suppose that a and b are distinct integers chosen from S. For how many ordered pairs (a, b) is $\log_a^b$ an integer?

Solution: $\log_{5^k} 5^n = 5^{\frac{n}{k}} \in \mathbb{Z}$ if and only if k divides n. When $k = 1$, $n = 2, \ldots, 10$; $k = 2$, $n = 4, 6, 8, 10$; $k = 3$, $n = 6, 9$; $k = 4$, $n = 8$; $k = 5$, $n = 10$. So there are $9 + 4 + 2 + 1 + 1 = 17$ pairs of $(5^k, 5^n)$, $1 \le k < n \le 10$, that can make $\log_{5^k} 5^n$ integers.

(a) 55

(b) 19

(c) 17

(d) 0

(e) 以上皆非 None of the above

17. 令 $f(x) = x^4 - 6x^2 + 8x$。下列敘述何者正確?
Let $f(x) = x^4 - 6x^2 + 8x$. Which of the following statements is correct?

Solution: Solving $f'(x) = 4x^3 - 12x + 8 = 0$ gets $x = 1$ (double root) or -2. Now, $f''(1) = 0$ and $f''(-2) > 0$. Thus, the local minimum is $f(-2) = -24$, and $(1, 3)$ is an inflection point.

(a) The local maximum of $f(x)$ is positive and the local minimum is negative.

(b) The local maximum of $f(x)$ is negative and the local minimum is positive.

(c) The local maximum of $f(x)$ is positive and there is no local minimum.

(d) The local minimum of $f(x)$ is negative and there is no local maximum.

(e) 以上皆非 None of the above

18. 下列何者等同於
Which of the following is the same as

$$\frac{1}{1+\cot^2 t}+\frac{1}{1+\tan^2 t}?$$

Solution: $1+\cot^2 t=\csc^2 t=\frac{1}{\sin^2 t}$, $1+\tan^2 t=\sec^2 t=\frac{1}{\cos^2 t}$. Thus, the answer is $\sin^2 t+\cos^2 t$ which is 1.

(a) 1
(b) $\sin t$
(c) $\cos t$
(d) $\tan t+\cot t$
(e) 以上皆非 None of the above

19. 對任意實數 x 及 y,$|x-1|+|x-y|+|y-2012|$ 的最小值為何?
What is the smallest value for $|x-1|+|x-y|+|y-2012|$ for all real numbers x and y?

Solution: From $|x-1|+|x-y|+|y-2012| \geq |(x-1)-(x-y)|+|y-2012| \geq |(y-1)-(y-2012)| = 2011$, and when $x=1$ and $y=0$, $|x-1|+|x-y|+|x-2012|=2011$, we see that 2011 is the smallest value.

(a) 0
(b) 2011
(c) 2012
(d) 2013
(e) 以上皆非 None of the above

20. 關於以下兩個圓的敘述, 何者為正確?
Which statement about the following two circles are true?

$$x^2+y^2=2y, \quad (x-1)^2+(y+2)^2=5.$$

Solution: The circle $x^2 + y^2 = 2y$ is the one $x^2 + (y-1)^2 = 1$, centered at $(0,1)$ with radius 1. The circle $(x-1)^2 + (y+2)^2 = 5$ is centered at $(1,-2)$ with center $\sqrt{5}$. As $(0,0)$ is on both circle, the circles intersect at one or more points. The distance between the two centers is $\sqrt{1+3^2} = \sqrt{10}$, which is smaller than the sum of the radius $1+\sqrt{5}$, there is more than one point of intersection.

(a) 它們不相交。They do not intersect at any point.

(b) 它們相切, 且一個包含了另一個。They are tangent to each other, and one is inside the other.

(c) 它們相切, 且兩個相互不包含。They are tangent to each other, and neither one is inside the other.

(d) 它們的交點超過一個。The intersect at more than one point.

(e) 以上皆非。None of the above.

21. 函數 $y = x^2$ 和 $y = \frac{1}{x^2+1}$ 的圖形的交點個數為
The number of points of intersection of the graphs of $y = x^2$ and $y = \frac{1}{x^2+1}$ is

Solution: Set $x^2 = \frac{1}{x^2+1}$ and we have $x^2(x^2+1) = 1$. Putting $u = x^2$, and solve $u^2 + u - 1 = 0$ to get $u = \frac{-1\pm\sqrt{5}}{2}$. From $u = x^2$ and x is real, we have $x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}$. Hence there are two points of intersection of the two graphs.

(a) 0

(b) 1

(c) 2

(d) 3

(e) 以上皆非 None of the above

22. 計算定積分:
Calculate the following definite integral:

$$\int_0^8 (5 - |x-7|)\, dx$$

Solution: $\int_0^8 (5-|x-7|)\, dx = \int_0^7 (5-(7-x))\, dx + \int_7^8 (5-(x-7))\, dx = (\frac{1}{2}x^2 - 2x)|_0^7 + (12x - \frac{1}{2}x^2)|_7^8 = \frac{49}{2} - 14 + (12 - \frac{15}{2}) = \frac{21}{2} + \frac{9}{2} = 15.$

(a) 15

(b) 16

(c) 30

(d) 32

(e) 以上皆非 None of the above

23. 袋中有 26 個球, 每一個有一個號碼寫在其表面上, 且每一個號碼剛剛好出現在兩個球上。如果連續由袋裡一個一個將球取出, 必需取出多少個才能保證一定會有同號的兩個球被取出來?
There are 26 balls in a bag, each has a distinct number on it, and each number appears exactly on two balls. If one continuously draws out balls blindly until a matching pair is drawn. How many balls must be taken out to guarantee having a matching pair?

Solution: One might be unlucky and have the first thirteen balls all have the different numbers on it. But then the 14th has to match one of them.

(a) 25

(b) 14

(c) 12

(d) 3

(e) 以上皆非 None of the above

24. 有隻青蛙在草皮上跳了兩次, 每次跳一公尺遠。假設青蛙每次跳的方向是獨立而且任意決定的, 並且各方向的選定機會是一致的。在兩次跳後, 青蛙的位置與原先位置相距至多一公尺的機率是多少?
A frog makes 2 jumps on a lawn, each 1 meter in length. The directions of the jumps are chosen independently and at random with equal chances for every direction to be chosen. What is the probability that the frogs final position is at most 1 meter from its starting position??

Solution: Suppose the frog starts at $(1,0)$ and his first jump is along the x-axis and that puts him at $(0,0)$. Then his second jump must be at angle θ where $-60 \leq \theta \leq 60$ in order for his final position to be less than 1 meter away from the original position. This range of angles is one third of the circle, hence the probability is one third.

(a) $\frac{2}{3}$

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{1}{4}$

(e) 以上皆非 None of the above

25. 令 $S = \{101, 102, 103, 104, 105, 106, 107, 108, 109\}$。假設由 S 中任取三個相異數, 其和為偶數的機率為 r。下列何者為真?

Let $S = \{101, 102, 103, 104, 105, 106, 107, 108, 109\}$. Let r be the probability that three random chosen distinct numbers from S have a sum an even number. Which of the following is true?

Solution: There are $\binom{9}{3} = 84$ different triples from S. There are $\binom{4}{3} = 4$ triples with all even numbers, and there are $\binom{5}{2} \cdot \binom{4}{1} = 40$ triples with two odd numbers and one even number. These are the triples get summed up to an even number. Thus, the probability is $\frac{4+40}{84} = \frac{11}{21}$.

(a) $0 \le r \le \frac{1}{4}$

(b) $\frac{1}{4} \le r \le \frac{1}{2}$

(c) $\frac{1}{2} \le r \le \frac{3}{4}$

(d) $\frac{3}{4} \le r \le 1$

(e) 以上皆非 None of the above